MAT1341 Intro to Linear Algebra

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Subspace test

1) contains zero vector
2) closed under addition
3) closed under scalar multiplication

→ all planes through the origin are subspaces

Definition of a span

span $\{v_1, v_2, \ldots, v_m\}$
$= \{a_1 v_1 + a_2 v_2 + \cdots + a_m v_m \mid a \in \mathbb{R}\}$
$=$ set of all linear combinations of $v_1, v_{2,\ldots,} v_m$

The big theorem about spans

- spans are subspaces (proof: apply the subspace test

How to check if two spans are equal

- test to see if all elements in each span are in the span of the other

Linear (in)dependence

Definition

Let $v_1, \ldots, v_m$ be elements of a vector space V.

a. $\{v_1, \ldots, v_m\}$ is linearly dependent (LD) if:
   $\exists\ \mathrm{a}_1, \ldots, \mathrm{a}_\mathrm{m} \in \mathbb{R}$, not all 0, such that
   $a_1 v_1 + \cdots + a_m v_m = 0$
b. $\{v_1, \ldots v_m\}$ is linearly independent (LI)if:
   $\{v_1, \ldots, v_m\}$ isn't LD <u>and</u>
   the only solution to $a_1 v_1 + \cdots + a_m v_m = 0$ is $\mathrm{a}_1 = \cdots = \mathrm{a}_\mathrm{m} = 0$

Notice:

→ any set containing $\{0\}$ is LD (**all subspaces are therefore LD**)
→ we can reduce any LD spanning set

Theorem relating linearly independent sets to spanning sets

- the size of any linearly independent set in $V \leq$ the size of any spanning set of V

Basis and dimension

Definition of **a basis**:

Let V be a vector space and $\{v_1, \ldots, v_m\}$ be a set of vectors in V. $\{v_1, \ldots, v_m\}$ is called a **basis** of V if it is linearly independent and it spans V.

- a basis is:
  - a linearly independent spanning set of V
  - the biggest possible LI set in V
  - the smallest possible spanning set in V

Theorem

If $\{v_1, \ldots, v_m\}$ and $\{w_1, \ldots, w_n\}$ are two bases for a vector space V, then $m = n$.

Dimension of V, denoted $\dim(V)$, is defined as the number of elements in any linearly independent spanning set (any basis) of V.

The size of any linearly independent set in V is:

- $\leq \dim(\mathrm{V})$
- $\leq$ size of any spanning set of V

and:

- a basis is the biggest possible LI set in V and the smallest possible spanning set of V

Solving linear systems

homogeneous linear system: all RHSs are 0
inhomogeneous linear system: not all RHSs are 0
degenerate equations: ie. $0 = 0, 8 = 0$
general solution: the set of all possible solutions of a system
consistent linear system: there's at least one solution to the system
inconsistent linear system: there's no solutions to the system

Homogenous systems are always consistent.

Theorem
Any **linear** system has either 0, 1, or ∞ solutions.

Matrices

Elementary Row Operators

1) Adding/subtracting a multiple of one row to another
2) Interchanging two rows
3) Multiply a row by a non-zero scalar

- any solution before an ERo will still be a solution after an ERo
  - ERos will not change the general solution
- every ERo is reversible

Row Echelon Form (REF) and Reduced Row Echelon Form (RREF)

Definition
A matrix (augmented or not) is in **row echelon form (REF)** if:
i) all zero rows are at the bottom
ii) the first non-zero entry in each row is a 1
iii) each leading 1 is to the right of the leading 1's in the row above

A matrix is in **reduced row echelon form (RREF)** if, in addition:
i) each leading 1 is the only non-zero entry in its column

Reading off a general solution

Rules

(1) Is there a row of type "0, … , 0 | $*$"?
Yes: $S = \emptyset$
No: Continue

(2) Each column of the coefficient matrix has a leading 1?
Yes: S has 1 element
No: S has infinitely many elements

(1) and (2) work with matrixes in REF. For (3) onwards, RREF is required:

(3) If S has 1 element, solution is the vector in the augmented column.

(4) If S has infinitely many elements, the variables corresponding to columns without leading 1s become free parameters (generally s and t). The remaining variable can now be expressed in terms of the free parameters.

Rank of matrices

Definition
The rank of a matrix A, denoted by $\text{rank}(A)$, is the number of leading 1s (pivots) in any REF of A.

Theorem

Let $[A|b]$ be an augmented matrix. Then:

a) 0 solutions (inconsistent) <=> rank(A) < rank(A|b)

b) 1 solution <=> rank(A)=rank(A|b) and rank(A)= # columns of A

c) infinitely many solutions <=> rank(A)=rank(A|b) and rank(A) < # columns of A

See ch. 13 of textbook for applications

Matrix multiplication

Example

$$\underset{A}{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}} \times \underset{B}{\begin{bmatrix} 5 & 2 \\ 1 & 3 \\ 4 & 2 \end{bmatrix}} = \begin{bmatrix} \square & \square \\ \square & \square \end{bmatrix}$$

to multiply two matrices, the # of columns in A must equal the # of rows in B

- dimension of product matrix = **# of rows in A by # of columns in B**
  i.e. in this example, A has 2 rows and B has 2 columns, so the product matrix is 2 rows by 2 columns
- each element of the product matrix has a unique address; [1,1], [1,2], [2,1], [2,2] (row, column)
- element [1,1] is equal to the dot product of the first row of A and the 1st column of B (=19)
  - every element [x,y] of the product matrix is equal to $\underbrace{A_x}_{\text{row x in A}} \cdot \underbrace{B_y}_{\text{column y in B}}$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 5 & 2 \\ 1 & 3 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 5(1)+2(1)+3(4) & 2(1)+2(3)+3(2) \\ 4(5)+5(1)+6(4) & 4(2)+5(3)+6(2) \end{bmatrix}$$

$$= \begin{bmatrix} 19 & 14 \\ 49 & 35 \end{bmatrix}$$

Key words: row x column

Properties:

Let A, B be matrices.

- each column of AB is a linear combination of the columns of A
- AB can be defined while BA not
- $AB \neq BA$ (it can, but generally does not) (ie. matrix multiplication is non-commutative)

Good properties

Let A, B, C be matrices and $k \in \mathbb{R}$ be a scalar. Then, whenever defined:

a) $(AB)C = A(BC)$
b) $A(B + C) = AB + AC$
c) $(B + C)A = BA + CA$
d) $k(AB) = (kA)B = A(kB)$
e) $(AB)^T = B^T A^T$
f) If A is m x n, then $I_m A = A$ and $AI_n = A$
g) If A is m x n, then $A \cdot 0_{nxp} = 0_{mxp}$

Matrices and linear systems

(i) $Ax = b$ is consistent <=> b is a linear combination of the columns of A
(ii) $Ax = 0$ has a unique solution (x=0)
<=> the columns of A are linearly independent
<=> $rank(A) = \#\ columns\ of\ A$

Spaces associated with matrices

Column space

The **column space** is defined as $Col(A) \coloneqq span\{C_1, C_2, \ldots, C_n\} = im(A) = \{Ax \mid x \in \mathbb{R}^n\}$
→ "the span of the columns of A", "the image of A"

Row space

The **row space** is defined as $Row(A) \coloneqq span\{r_1, \ldots r_m\}$.

Null space

The **null space** or **kernel** of A is defined as $Null(A) \coloneqq \{x \in \mathbb{R}^n | \; Ax = 0\} = \ker(A)$

Rank-Nullity theorem

$$\dim\big(Null(A)\big) + rank(A) = \#\, of\ columns\ of\ A$$

Theorem

If $Ax = b$ is consistent and v is a particular solution that $Av = b$, then the general solution is:

$$S = v + Null(A)$$

Summary of facts

a) consistency of linear systems

Let $A \in \mathrm{M}_{mn}(\mathbb{R})$ and $b \in \mathbb{R}^m$.

Linear system $Ax = b$ is **consistent**
<=>$b \in$ Col(A)
<=>$rank(A) = rank(A|b)$

<=>every $b \in \mathbb{R}^m$ is in Col(A)
<=>Col(A)=$\mathbb{R}^m$
<=>rank(A)=m

b) number of solutions

Let $A \in \mathrm{M}_{mn}(\mathbb{R})$ and $b \in \mathbb{R}^m$ so that $Ax = b$ is consistent.
Linear system $Ax = b$ has a **unique solution**:
<=>no free parameters in the general solution
<=>every column of the RREF of A has a leading one
<=>linear system $Ax = 0$ has the unique solution $x = 0$
<=>columns of A are linearly independent
<=>Null(A)= {0}
<=>dim(Null(A))= 0
<=>rank(A)=n *(see rank-nullity theorem)

Note

We can solve $Ax = b$ for every b **iff** rank(A)=m (infinite solutions). Such a solution is unique **iff** rank(A)=n.

Finding bases

Row space algorithm

1) Write vectors as rows into matrix
2) Transform into any REF (or into RREF if you like)
3) Non-zero rows form a basis of W

Extending linearly independent sets to a basis of $\mathbb{R}^n$

Use the row space algorithm

1) Write vectors as rows into matrix.
2) Transform into any REF (or RREF).
3) If the k-th column has no leading one, add the vector (0 … 1 … 0) to the LI set. This yields a basis of $\mathbb{R}^n$.

Reducing spanning sets to a basis of W

Use the column space algorithm

1) Write vectors in columns of matrix
2) Transform into any REF (or RREF).
3) Keep the vectors that correspond to columns with leading ones.

And so:

$\dim(Row(A)) = rank(A)$
$\dim(Col(A)) = rank(A)$
$\dim(Row(A)) = \dim(Col(A))$
$\dim(Null(A)) = \#\text{ of columns of } A - rank(A)$

Finding bases in general vector spaces

Problem:

Find a basis of the subspace W of $\mathbb{P}_3$:
$W = span\{3 + x + 4x^2 + 2x^3, 2 + 4x + 6x^2 + 8x^3, 1 + 3x + 4x^2 + 6x^3, -1 + 2x + x^2 + 4x^3\}$

Choose an ordered basis of $\mathbb{P}_3$, say $B = \{1, x, x^2 x^3\}$, and work with the coordinate vectors, which live in $\mathbb{R}^4$.

$$span\left\{\begin{bmatrix}3\\1\\4\\2\end{bmatrix}_B, \begin{bmatrix}2\\4\\6\\8\end{bmatrix}_B, \begin{bmatrix}1\\3\\4\\6\end{bmatrix}_B, \begin{bmatrix}-1\\2\\1\\4\end{bmatrix}_B\right\}$$

Run the row space algorithm into RREF...

Read off a basis and translate back to $\mathbb{P}_3$:

$$\left\{\begin{bmatrix}1\\0\\1\\0\end{bmatrix}_B, \begin{bmatrix}0\\1\\1\\2\end{bmatrix}_B\right\} \rightarrow \{1 + x^2, x + x^2 + 2x^3\}$$

Extending LI sets to bases in general vector spaces

Problem:

Extend $\{1 + x^2, x + x^2 + 2x^3\}$ to a basis of $\mathbb{P}_3$. We use the same strategy as before.

Consider $\left\{\begin{bmatrix}1\\0\\1\\0\end{bmatrix}_B, \begin{bmatrix}0\\1\\1\\2\end{bmatrix}_B\right\}$ and add two more vectors:

$$\begin{bmatrix}1 & 0 & 1 & 0\\0 & 1 & 1 & 2\\0 & 1 & 0 & 0\\0 & 0 & 0 & 1\end{bmatrix} \sim \begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix}$$

The basis of $\mathbb{R}^4$ is:

$$\left\{\begin{bmatrix}1\\0\\1\\0\end{bmatrix}_B, \begin{bmatrix}0\\1\\1\\2\end{bmatrix}_B, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}_B, \begin{bmatrix}0\\0\\0\\1\end{bmatrix}_B\right\} \rightarrow \{1 + x^2, x + x^2 + 2x^3, x, x^3\}$$

Invertible matrices

Only for nxn matrices:

rank(A)=n (# of rows and # of columns)
<=> "the columns form a basis of $\mathbb{R}^n$"
<=> $Ax = b$ is consistent (because rank(A)=n) <u>and</u> has a unique solution for every $b \in \mathbb{R}^n$ (because rank(A)=n=m)

Definition

Let A be a nxn matrix.
If B is also an nxn matrix such that $AB = I$ and $BA = I$, then B is called an inverse of A, and A is called **invertible**.

Finding inverses of 2x2 matrices

Lemma: Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then:

1) if $ad - bc \neq 0$, then $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$
2) if $ad - bc = 0$, then A is not invertible.

ie. first check if *ad*-*bc*=0, and if so then use the formula.

Algebraic properties of invertible matrices

Let $k \neq 0$ be a scalar, p an integer, A and C invertible *nxn* matrices.
Then, the following are also invertible:

1) A^{-1} $\quad (A^{-1})^{-1} = A$
2) A^p $\quad (A^p)^{-1} = (A^{-1})^p$
3) A^T $\quad (A^T)^{-1} = (A^{-1})^T$
4) kA $\quad (kA)^{-1} = \frac{1}{k}A^{-1}$
5) AC $\quad (AC)^{-1} = C^{-1}A^{-1}$

also: if AB=I, then BA=I.

Invertible matrix theorem

Let A be an *nxn* matrix. Then, the following statements are equivalent:

1) **A is invertible**
2) $rank(A) = n$ (there are n leading ones; for every column in the REF there's a leading 1)
3) $Ax = 0$ has only the trivial solution ($x = 0$)
4) $Ax = b$ is consistent for every $b \in \mathbb{R}^n$
5) $Ax = b$ has a unique solution for every $b \in \mathbb{R}^n$
6) The RREF of A is I (the identity matrix)
7) Null(A)= $\{x \in \mathbb{R}^n \mid Ax = 0\} = \{0\}$
8) Col(A)= $\{Ax \mid x \in \mathbb{R}^n\} = \mathbb{R}^n$
9) Row(A)=$\mathbb{R}^n$
10) $\text{rank}(A^T) = n$

    proof:

    $$\begin{aligned} &\text{rank}(A) \\ &= \dim(Row(A)) \\ &= \dim(Col(A)) \\ &= \dim(Row(A^T)) \\ &= rank(A^T) \end{aligned}$$

11) The columns of A are linearly independent.
12) The rows of A are linearly independent.
13) The columns of A span $\mathbb{R}^n$
14) The rows of A span $\mathbb{R}^n$
15) The columns of A are a basis of $\mathbb{R}^n$
16) The rows of A are a basis of $\mathbb{R}^n$
17) A^T is invertible.
18) det(A)$\neq 0$

Orthogonality

A set of vectors $\{v_1, \dots, v_m\} \subseteq \mathbb{R}^n$ is called **orthogonal** if:

(1) $v_i \cdot v_j = 0$ for $1 \leq i \leq j \leq m$

(2) $v_i \neq 0$ for $1 \leq i \leq m$

the set is called **orthonormal** if, in addition to (1) and (2):

(3) $v_i \cdot v_i = ||v_i||^2 = 1$ for all $1 \leq i \leq m$

Theorem

Orthogonal sets are linearly independent.

But linearly independent sets are not necessarily orthogonal.

Corollary
An orthogonal spanning set is a basis

Orthogonal projections
Let W be a subspace of $\mathbb{R}^n$ with orthogonal basis $\{w_1, \dots, w_n\}$.
Then, for any vector v in $\mathbb{R}^n$ (in the subspace or not), we can set:

$$proj_W(v) = \frac{w_1 \cdot v}{w_1 \cdot w_1} \cdot w_1 + \cdots + \frac{w_m \cdot v}{w_m \cdot w_m} \cdot w_m$$

"the orthogonal projection of v onto w"

Gram-Schmidt algorithm
Let $\{u_1, \dots u_m\}$ be a basis of a subspace U of $\mathbb{R}^n$.
Then, define:

$w_i := u_1$

$w_2 := u_2 - proj_{v_1}(u_2)$

$= u_2 - \frac{w_1 \cdot u_2}{w_1 \cdot w_1} \cdot w_1$

$w_3 := u_3 - proj_{v_2}(u_3)$ where $V_2 = span\{u_1, u_2\}$

$$= u_3 - \frac{w_1 \cdot u_3}{w_1 \cdot w_1} \cdot w_1 - \frac{w_2 \cdot u_3}{w_2 \cdot w_2} \cdot w_2$$

$\vdots$

$w_m := u_m - proj_{v_{m-1}}(u_m)$

$$= u_m - \frac{w_1 \cdot u_m}{w_1 \cdot w_1} \cdot w_1 - \cdots - \frac{w_{(m-1)} \cdot u_m}{w_{m-1} \cdot w_{m-1}} \cdot w_{m-1}$$

This, $\{w_1, \dots, w_m\}$ is an orthogonal basis for U.

Determinants

Definition
There are 3 cases with respect to the determinant:

$n = 1$ $\quad A = [a]$ $\quad |A| = \det(A) = a$

$n = 2$ $\quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\quad \det(A) = ad - bc$

$n \geq 3$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & \cdots & a_{2n} \\ \vdots & \vdots & . & . & \vdots \\ \vdots & \vdots & . & . & \vdots \\ a_{n1} & a_{n2} & \cdots & \cdots & a_{nn} \end{bmatrix}$$

Let A_{ij} be the $(n-1)x(n-1)$ matrix obtained from A by removing the i-th row and the j-th column.

<u>Alternative 1:</u>
Pick the i-th row of A, whatever i you like best.

$\det(A)$
$= (-1)^{i+1} \cdot a_{i1} \cdot \det(A_{i1}) + (-1)^{i+2} \cdot a_{i2} \cdot \det(A_{i2}) + \cdots$
$+ (-1)^{i+n} \cdot \mathrm{a_{in}} \cdot \det(\mathrm{A_{in}})$

<u>Alternative 2:</u>
Pick the j-th column of A, whatever j you like best.

$\det(A)$
$= (-1)^{1+j} \cdot a_{1j} \cdot \det(A_{1j}) + (-1)^{2+j} \cdot a_{2j} \cdot \det(A_{2j})$
$+ \cdots + (-1)^{n+j} \cdot \mathrm{a_{nj}} \cdot \det(\mathrm{A_{nj}})$

When computing the determinant, choose to expand along the row or column with the most number of zeroes.

Properties
1) If A has a row or column of zeroes, then $\det(A) = 0$
2) $\det(A^T) = \det(A)$
3) If you have a triangular matrix (either all entries above or below the diagonal are

zero), then $\det(A)$ is the product of the diagonal entries.

4) $\det(rA) = r^n \cdot \det(A)$
5) $\det(AB) = \det(A) \cdot \det(B)$
6) $\det(A) \neq 0 \Rightarrow A \text{ is invertible}$

Determinants and Elementary Row Operations

interchange 2 rows	$\det(B) = -\det(A)$
multiply one row by a scalar	$\det(B) = r \cdot \det(A)$
add a multiple of one row to another	$\det(B) = \det(\text{A})$

Eigenvalues and Eigenvalues

Definition

Let A be an nxn matrix.

If $\lambda \in \mathbb{R}$ and $x \in \mathbb{R}^n, x \neq 0$
such that:

$$Ax = \lambda x$$

then λ is called an **eigenvalue** of A and x is called an **eigenvector** of A.

How to find eigenvalues

- compute the **characteristic polynomial**, $\det(A - \lambda I)$, of A and solve for λ - these ar e the eigenvalues

How to find eigenvectors

- compute $E_\lambda = \ker(A - \lambda I)$, ie. the **eigenspace**
- the basis of the eigenspace is/are the eigenvector(s)

Diagonizability

An *nxn* matrix A is called "**diagonizable**" if there's a basis of $\mathbb{R}^n$ consisting entirely of eigenvectors in A.

Final conclusions about diagonizability

The multiplicity of λ as a root of the characteristic polynomial is called the **algebraic multiplicity** of λ. Moreover, $\dim(E_\lambda)$ is called the **geometric multiplicity** of λ.

Theorem

Let λ be an eigenvalue. Then, $1 \leq geom.multiplicity \leq alg.multiplicity$.

Theorem

Eigenvectors corresponding to different eigenvalues are linearly independent.

Corollary

1) If an nxn matrix has n distinct eigenvalues, it is diagonizable.
2) If $\lambda_1, \dots, \lambda_k$ are the distinct eigenvalues of A, then:
   A diagonizable
   <=> the sums of the geom. multiplicities is n
   <=> $\dim(E_{\lambda_1}) + \cdots + \dim(E_{\lambda_n}) = n$

What to do to decide about diagonizability?

1) Compute $\det(A - \lambda I)$, find all roots.
   This requires a factorization into linear factors. If that's impossible, A is not diagonizable over the reals.
2) Find a basis for each eigenspace.
3) If there are *n* vectors in all of the these bases,

→ diagonizable

If there are fewer than n vectors in all of these bases,

→ not diagonizable

4) In the case of diagonizability, write the basis vectors into the columns of P. Then,

$$P^{-1}AP = D$$

(P is the matrix with the eigenvectors of A as its columns, while D is the matrix with diagonals as eigenvalues corresponding to the position of the eigenvectors in P)